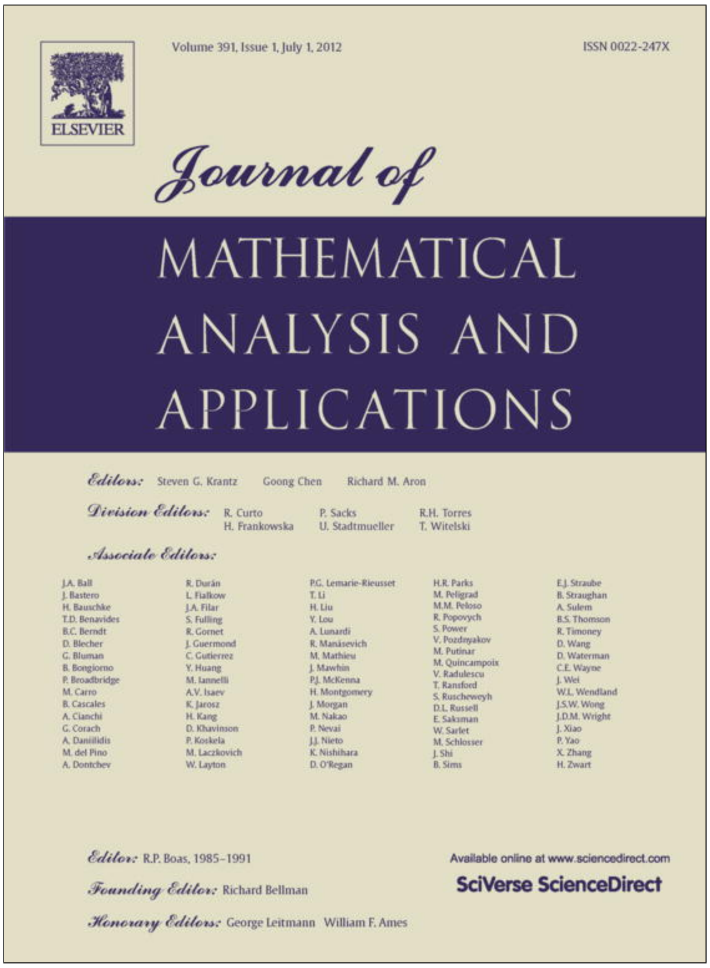
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# Global nonexistence for a semilinear wave equation with nonlinear boundary dissipation ☆

Hongyinping Feng [a,*], Shengjia Li [b]

[a] School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, PR China
[b] Research Institute of Mathematics and Applied Mathematics, Shanxi University, Taiyuan, Shanxi, 030006, PR China

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ABSTRACT

In this paper, we consider the following problem

$$\begin{cases} y_{tt} - y_{xx} + y_t = |y|^{p-1}y, & (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, & t \in (0,T), \\ y_x(L,t) + y(L,t) + \left|y_t(L,t)\right|^{m-1} y_t(L,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), & x \in (0,L), \end{cases}$$

where $(0,L)$ is a bounded open interval in $\mathbb{R}$, $p > 1$ and $m \geqslant 1$. We are interested in the interaction between the boundary damping $|y_t(L,t)|^{m-1} y_t(L,t)$ and the interior source $|y(t)|^{p-1}y(t)$. Under some appropriate assumptions on the initial data, two blow-up results with positive initial energy are established. Furthermore, we obtain that the solutions blow up if $p$ or $|y^0(L)|$ is sufficiently large.



## 1. Introduction

We study the following initial boundary value problem:

$$\begin{cases} y_{tt} - y_{xx} + y_t = |y|^{p-1}y, & (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, & t \in (0,T), \\ y_x(L,t) + y(L,t) + \left|y_t(L,t)\right|^{m-1} y_t(L,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), & x \in (0,L), \end{cases} \tag{1.1}$$

where $(0,L)$ is a bounded open interval in $\mathbb{R}$, $p > 1$ and $m \geqslant 1$.

In recent years, there is a large body of literature regarding the interaction between the damping terms and the source terms. For the interaction between the internal damping and the internal source, we refer the reader to [1–7]. For the interaction between the boundary damping and the boundary source, we refer the reader to [8–14].

When the damping terms are absent, it is well known that the source terms cause blow-up of solutions in finite time for sufficiently large initial data. When source terms are absent, the damping terms cause global existence for arbitrary initial

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* Corresponding author. Fax: +86 351 7010979.
*E-mail address:* fhyp@sxu.edu.cn (H. Feng).

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data. In [15], E. Vitillaro obtained that the solution of problem (1.1) blows up if $m < \frac{4(p+1)}{p+3}$ (we only consider $n = 1$) and the initial data is inside the unstable set which is obtained by the potential well theory. However, he did not consider the case of $m \geqslant \frac{4(p+1)}{p+3}$. For the interaction between the boundary damping and the internal source, we refer the reader to [10,11,16].

In this paper, we consider the global nonexistence for problem (1.1) whether $m < \frac{4(p+1)}{p+3}$ or not. Under some appropriate assumptions on the initial data, we obtain that the solution blows up if $p$ or $|y^0(L)|$ is sufficiently large.

The paper is organized as follows. In Section 2, we introduce some notations and state the main results. In Section 3, we prepare several lemmas which will be applied to the proofs of Theorems 2.1 and 2.2. Sections 4 and 5 are devoted to the proofs of Theorems 2.1 and 2.2 respectively.

The energy of problem (1.1) is defined by

$$E(t) = \frac{1}{2}\|y_t(t)\|_2^2 + \frac{1}{2}\|y_x(t)\|_2^2 + \frac{1}{2}|y(L,t)|^2 - \frac{1}{p+1}\|y(t)\|_{p+1}^{p+1}. \tag{1.2}$$

In order to state the local existence theorem for problem (1.1), we define

$$H_R^1 := \{v \in H^1(0,L)\colon\ v(0) = 0\}. \tag{1.3}$$

The following local existence and regularity result for the problem (1.1) can be obtained by the standard Galerkin approximation. So we omit the details.

**Theorem 1.1** *(Local solution). Assume that* $(y^0, y^1) \in H_R^1 \times L^2(0,L)$. *Then problem* (1.1) *has a unique local solution* $y$ *satisfying*

$$y(x,t) \in C(0,T_0; H_R^1), \qquad y_t(x,t) \in C(0,T_0; L^2(0,L)) \tag{1.4}$$

*for some* $T_0 > 0$, *and energy identity*

$$E(t) + \int_0^t |y_t(L,\tau)|^{m+1}\,d\tau + \int_0^t \|y_t(\tau)\|_2^2\,d\tau = E(0) \tag{1.5}$$

*holds for* $0 \leqslant t \leqslant T_0$.

By taking a derivative of (1.5), we get

$$\frac{d}{dt}E(t) = -|y_t(L,t)|^{m+1} - \|y_t(t)\|_2^2. \tag{1.6}$$

This means that the energy of problem (1.1) is dissipative.

## 2. Notations and main results

Throughout this paper, $C$ and $C_1$ denote generic constants.

In order to state our main results we define

$$E_1 := \left(\frac{1}{2} - \frac{1}{p+1}\right)\alpha_0, \qquad \alpha_0 := B^{-\frac{2(p+1)}{p-1}}, \tag{2.1}$$

where $B$ is the optimal constant of the Sobolev embedding $\|v\|_{p+1} \leqslant B\|v_x\|_2$ for $v \in H_R^1$, and

$$J(t) := \|y_x(t)\|_2^2 - \|y(t)\|_{p+1}^{p+1}. \tag{2.2}$$

For the convenience of the reader, we introduce a function

$$\eta(\zeta) := \frac{2}{p+1-\zeta(p-1)} \quad \text{for all } \zeta \in \mathbb{R}. \tag{2.3}$$

Now we are in position to state our main results.

**Theorem 2.1.** *Let* $y(x,t)$ *be a solution of problem* (1.1). *Assume that* $J(0) < 0$ *and* $E(0) = \lambda E_1$ *for some constant* $0 < \lambda < 1$. *Assume further that*

$$|y^0(L)| > \frac{2\mathcal{M} + 1 + \eta(\lambda)(p-1)(\mathcal{M}+1)/2}{[1-\eta(\lambda)](p-1)}, \tag{2.4}$$

*where*

$$M := \sup\{|y^0(x)|\colon\ x \in [0,L]\}, \qquad \mathcal{M} := \sup\{|y_x^0(x)|\colon\ x \in [0,L]\}. \tag{2.5}$$

*Then the solution blows up in finite time.*

**Remark 2.1.** Since (2.4) holds true if $|y^0(L)|$ is large enough, we can conclude that, in addition to $0 < E(0) < E_1$, the solution always blows up whenever $|y^0(L)|$ is sufficiently large.

**Remark 2.2.** Since

$$\lim_{p\to+\infty}\left[\frac{2\mathcal{M}+1+\eta(\lambda)(p-1)(\mathcal{M}+1)/2}{[1-\eta(\lambda)](p-1)}\right]=0, \tag{2.6}$$

(2.4) holds true whenever $p$ is large enough. So we can conclude that, in addition to $0 < E(0) < E_1$, the blow-up result is independent of the boundary damping $|y_t(L,t)|^{m-1}y_t(L,t)$, provided $p$ is chosen large enough.

**Theorem 2.2.** *Let $y(x,t)$ be a solution of problem* (1.1)*. Assume that $J(0) < 0$ and $E(0) = \lambda E_1$ for some constant $0 < \lambda < 1$. Assume further that*

$$L > \frac{3+(p-1)\eta(\lambda)}{(p-1)[1-\eta(\lambda)]}. \tag{2.7}$$

*Then the solution blows up in finite time.*

**Remark 2.3.** Since (2.7) holds true if $L$ is large enough, we can conclude that, in addition to $0 < E(0) < E_1$, the blow-up result is independent of the boundary damping $|y_t(L,t)|^{m-1}y_t(L,t)$, provided $L$ is chosen large enough.

**Remark 2.4.** Since

$$\lim_{p\to+\infty}\left[\frac{3+(p-1)\eta(\lambda)}{(p-1)[1-\eta(\lambda)]}\right]=0, \tag{2.8}$$

(2.7) holds true whenever $p$ is large enough. So we can conclude that, in addition to $0 < E(0) < E_1$, the solution always blows up whenever $p$ is sufficiently large.

## 3. Some preliminary results

In this section, we put forward some preliminary lemmas, which will be applied to the proofs of the theorems.

**Lemma 3.1.** *Let $y(x,t)$ be a solution of problem* (1.1)*. Assume that $J(0) < 0$ and $E(0) = \lambda E_1$ for some constant $0 < \lambda < 1$. Then*

$$0 < \eta(\lambda) < 1 \tag{3.1}$$

*and*

$$2E_1 < \eta(\lambda)\frac{p-1}{p+1}\|y(t)\|_{p+1}^{p+1}, \quad \forall t > 0. \tag{3.2}$$

**Proof.** Form (1.2) and the Sobolev embedding, we have

$$E(t) \geqslant \frac{1}{2}\|y_x(t)\|_2^2 - \frac{1}{p+1}\|y(t)\|_{p+1}^{p+1} \geqslant \frac{1}{2}\|y_x(t)\|_2^2 - \frac{B^{p+1}}{p+1}\|y_x(t)\|_2^{p+1}.$$

Now if we let $g(\xi) = \frac{1}{2}\xi - \frac{B^{p+1}}{p+1}\xi^{\frac{p+1}{2}}$, then

$$E(t) \geqslant g(\xi) \quad \text{with} \quad \xi = \|y_x(t)\|_2^2. \tag{3.3}$$

It is easy to verify that the function $g(\xi)$ has the following properties:

$$\begin{cases} g(\xi) \text{ is strictly increasing on } [0,\alpha_0), \\ g(\xi) \text{ takes its maximum value } E_1 \text{ at } \alpha_0, \\ g(\xi) \text{ is strictly decreasing on } (\alpha_0,+\infty). \end{cases} \tag{3.4}$$

It follows from the fact $J(0) < 0$ that

$$\|y_x^0\|_2^2 < |y^0(L)|^{p+1} \leqslant B^{p+1}\|y_x^0\|_2^{p+1}. \tag{3.5}$$

Consequently,

$$\|y_x^0\|_2^2 > \alpha_0. \tag{3.6}$$

Since

$$E_1 > E(0) \geqslant E(t) \geqslant g\big(\|y_x(t)\|_2^2\big) \tag{3.7}$$

for all time $t \geqslant 0$, it follows from (3.4) and (3.7) that there is no time $t^*$ such that $\|y_x(t^*)\|_2^2 = \alpha_0$. By the continuity of $\|y_x(t)\|_2^2$ with respect to the time variable and (3.6), we have

$$\|y_x(t)\|_2^2 > \alpha_0, \quad \forall t \geqslant 0. \tag{3.8}$$

Consequently,

$$\frac{1}{p+1}\|y(t)\|_{p+1}^{p+1} \geqslant -E(0) + \frac{1}{2}\|y_t(t)\|_2^2 + \frac{1}{2}\|y_x(t)\|_2^2 > -\lambda E_1 + \frac{1}{2}\alpha_0 = \left(\frac{p+1}{p-1} - \lambda\right)E_1. \tag{3.9}$$

Combining (2.3) and (3.9), we can conclude that inequalities (3.1) and (3.2) hold true. The proof is complete. □

**Lemma 3.2.** *Let $y(x,t)$ be a solution of problem (1.1). Assume that $0 < E(0) < E_1$ and $2 \leqslant s \leqslant p+1$. Then, there exists a constant $C > 0$ such that*

$$\|y\|_{p+1}^s \leqslant C\|y\|_{p+1}^{p+1}. \tag{3.10}$$

**Proof.** If $\|y\|_{p+1} \geqslant 1$ then $\|y\|_{p+1}^s \leqslant \|y\|_{p+1}^{p+1}$. If $\|y\|_{p+1} < 1$, it follows from the Sobolev embedding that

$$\|y\|_{p+1}^s \leqslant \|y\|_{p+1}^2 \leqslant C\|y_x\|_2^2. \tag{3.11}$$

From Lemma 3.1 we have

$$\|y_x\|_2^2 \leqslant 2E_1 + \frac{2}{p+1}\|y\|_{p+1}^{p+1} \leqslant \|y\|_{p+1}^{p+1}. \tag{3.12}$$

This completes the proof. □

**Lemma 3.3.** *Assume that there exists a function $N(t)$ satisfying $N(0) > 0$ and*

$$\frac{d}{dt}N(t) \geqslant \gamma\big[|y_t(L,t)|^{2m} + |y(L,t)|^{p+1}\big] \quad \textit{for all } t \geqslant 0, \tag{3.13}$$

*where $\gamma$ is a positive constant. Assume further that, there exists a constant $C > 0$, such that*

$$N(t) \leqslant C\|y(t)\|_{p+1}^{p+1}, \quad \forall t \geqslant 0. \tag{3.14}$$

*Then, the solution blows up in finite time.*

**Proof.** Define

$$Q(t) := N^{1-\alpha}(t) + \theta\int_0^L y_t(t)y(t)\,dx + \frac{\theta}{2}\|y(t)\|_2^2 \quad \text{with} \quad \alpha = \frac{p-1}{2(p+1)}, \tag{3.15}$$

where $\theta$ is small, to be determined later.

By taking a derivative of $Q(t)$, we have

$$\begin{aligned}\frac{d}{dt}Q(t) \geqslant{}& (1-\alpha)N^{-\alpha}(t)\gamma\big[|y_t(L,t)|^{2m} + |y(L,t)|^{p+1}\big] - 2\theta E_1 \\ &+ 2\theta\|y_t(t)\|_2^2 + 2\theta\big[E_1 - E(t)\big] - \theta|y_t(L,t)|^m|y(L,t)| + \theta\frac{p-1}{p+1}\|y(t)\|_{p+1}^{p+1}.\end{aligned} \tag{3.16}$$

By Young's inequality, we have

$$|y_t(L,t)|^m|y(L,t)| = \left[\frac{|y_t(L,t)|^m|y(L,t)|}{\delta}\right]\cdot\delta \leqslant (1-\alpha)\delta^{-\frac{1}{1-\alpha}}\big[|y_t(L,t)|^m|y(L,t)|\big]^{\frac{1}{1-\alpha}} + \alpha\delta^{\frac{1}{\alpha}} \tag{3.17}$$

for $\delta > 0$. If we let $\delta^{1/\alpha} = K^{-1/\alpha}\cdot N^{1-\alpha}(t)$, i.e., $\delta^{-1/(1-\alpha)} = K^{1/(1-\alpha)}\cdot N^{-\alpha}(t)$, $K > 0$ to be determined later, then

$$\frac{d}{dt}Q(t) \geqslant (1-\alpha)N^{-\alpha}(t)\left\{\gamma\left[\left|y_t(L,t)\right|^{2m} + \left|y(L,t)\right|^{p+1}\right] - \theta K^{\frac{1}{1-\alpha}}\left[\left|y_t(L,t)\right|^m\left|y(L,t)\right|\right]^{\frac{1}{1-\alpha}}\right\}$$
$$+ 2\theta\left\|y_t(t)\right\|_2^2 - \theta\alpha K^{-1/\alpha}\cdot N^{1-\alpha}(t) + \left\{\left[1-\eta(\lambda)\right]\theta\frac{p-1}{p+1}\right\}\left\|y(t)\right\|_{p+1}^{p+1}. \tag{3.18}$$

Again we use Young's inequality

$$XY \leqslant \frac{X^r}{r} + \frac{Y^s}{s}, \quad X, Y \geqslant 0 \quad \text{with} \quad \frac{1}{r} + \frac{1}{s} = 1 \tag{3.19}$$

for $|y_t(L,t)|^{\frac{m}{1-\alpha}} \cdot |y(L,t)|^{\frac{1}{1-\alpha}}$ with $r = 2(1-\alpha)$; then $\frac{s}{1-\alpha} = \frac{2}{1-2\alpha} = p+1$, so we have

$$\frac{d}{dt}Q(t) \geqslant (1-\alpha)N^{-\alpha}(t)\left(\gamma - \theta C K^{\frac{1}{1-\alpha}}\right)\left[\left|y_t(L,t)\right|^{2m} + \left|y(L,t)\right|^{p+1}\right]$$
$$+ 2\theta\left\|y_t(t)\right\|_2^2 - \theta\alpha K^{-1/\alpha}\cdot N^{1-\alpha}(t) + \left\{\left[1-\eta(\lambda)\right]\theta\frac{p-1}{p+1}\right\}\left\|y(t)\right\|_{p+1}^{p+1}. \tag{3.20}$$

It follows from (3.14), (3.15) and Lemma 3.2 that

$$N^{1-\alpha}(t) \leqslant C_1\left\|y(t)\right\|_{p+1}^{(1-\alpha)(p+1)} \leqslant C_1\left\|y(t)\right\|_{p+1}^{p+1}. \tag{3.21}$$

Moreover, we have

$$\frac{d}{dt}Q(t) \geqslant (1-\alpha)N^{-\alpha}(t)\left(\gamma - \theta C K^{\frac{1}{1-\alpha}}\right)\left[\left|y_t(L,t)\right|^{2m} + \left|y(L,t)\right|^{p+1}\right]$$
$$+ 2\theta\left\|y_t(t)\right\|_2^2 + \left\{\left[1-\eta(\lambda)\right]\theta\frac{p-1}{p+1} - \theta\alpha C_1 K^{-1/\alpha}\right\}\left\|y(t)\right\|_{p+1}^{p+1}. \tag{3.22}$$

At this point, we choose $K$ large enough so that (3.22) becomes

$$\frac{d}{dt}Q(t) \geqslant (1-\alpha)N^{-\alpha}(t)\left(\gamma - \theta C K^{\frac{1}{1-\alpha}}\right)\left[\left|y_t(L,t)\right|^{2m} + \left|y(L,t)\right|^{p+1}\right] + \theta\rho\left[\left\|y_t(t)\right\|_2^2 + \left\|y(t)\right\|_{p+1}^{p+1}\right], \tag{3.23}$$

where $\rho > 0$ is the minimum of coefficients of $\|y_t(t)\|_2^2$ and $\|y(t)\|_{p+1}^{p+1}$. Once $K$ is fixed, we pick $\theta$ small enough so that

$$\gamma - \theta C K^{\frac{1}{1-\alpha}} \geqslant 0 \tag{3.24}$$

and

$$Q(0) = N^{1-\alpha}(0) + \theta\int_0^L y^0 y^1\,dx + \frac{\theta}{2}\left\|y^0\right\|_2^2 > 0. \tag{3.25}$$

Therefore, (3.23) takes the form

$$\frac{d}{dt}Q(t) \geqslant \theta\rho\left[\left\|y_t(t)\right\|_2^2 + \left\|y(t)\right\|_{p+1}^{p+1}\right]. \tag{3.26}$$

On the other hand, by Hölder's inequality, Young's inequality and the Sobolev embedding, we have

$$\left|\int_0^L y(t)y_t(t)\,dx\right|^{1/(1-\alpha)} \leqslant C\left\|y(t)\right\|_{p+1}^{1/(1-\alpha)}\left\|y_t(t)\right\|_2^{1/(1-\alpha)} \leqslant C\left[\left\|y(t)\right\|_{p+1}^{p+1} + \left\|y_t(t)\right\|_2^2\right]. \tag{3.27}$$

Similarly, we have

$$\left[\frac{\varepsilon}{2}\left\|y(t)\right\|_2^2\right]^{1/(1-\alpha)} \leqslant C\left|\int_0^L y(t)y(t)\,dx\right|^{1/(1-\alpha)} \leqslant C\left[\left\|y(t)\right\|_{p+1}^{p+1} + \left\|y(t)\right\|_2^2\right]. \tag{3.28}$$

From (1.2) and Lemma 3.1 we have

$$\left\|y_x(t)\right\|_2^2 \leqslant 2E_1 + \frac{2}{p+1}\left\|y(t)\right\|_{p+1}^{p+1} \leqslant \left\|y(t)\right\|_{p+1}^{p+1}. \tag{3.29}$$

Using the Sobolev embedding, we have

$$\left[\frac{\varepsilon}{2}\|y(t)\|_2^2\right]^{1/(1-\alpha)} \leqslant \left[\|y(t)\|_{p+1}^{p+1} + \|y_x(t)\|_2^2\right] \leqslant C\|y(t)\|_{p+1}^{p+1}. \tag{3.30}$$

It follows from (3.14), (3.27) and (3.30) that

$$\begin{aligned} Q^{1/(1-\alpha)}(t) &= \left[N^{1-\alpha}(t) + \theta\int_0^L y(t)y_t(t)\,dx + \frac{\theta}{2}\|y(t)\|_2^2\right]^{1/(1-\alpha)} \\ &\leqslant C\left\{N(t) + \left|\int_0^L y(t)y_t(t)\,dx\right|^{1/(1-\alpha)} + \left[\|y(t)\|_2^2\right]^{1/(1-\alpha)}\right\} \\ &\leqslant C\left[\|y_t(t)\|_2^2 + \|y(t)\|_{p+1}^{p+1}\right]. \end{aligned} \tag{3.31}$$

Combining (3.26) and (3.31), we arrive at

$$\frac{d}{dt}Q(t) \geqslant \Gamma Q^{1/(1-\alpha)}(t), \quad \forall t \geqslant 0, \tag{3.32}$$

where $\Gamma$ is a positive constant depending only on $\theta\rho$ and $C$. Integrating (3.32) over $(0,t)$ then yields

$$Q^{\alpha/(1-\alpha)}(t) \geqslant \frac{1}{Q^{-\alpha/(1-\alpha)}(0) - \frac{\Gamma\alpha}{1-\alpha}t}. \tag{3.33}$$

Therefore, (3.33) shows that $Q(t)$ blows up in time

$$T^* \leqslant \frac{1-\alpha}{\Gamma\alpha[Q(0)]^{\alpha/(1-\alpha)}}. \tag{3.34}$$

This completes the proof. □

## 4. The proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. If we let

$$\mathrm{Sgn}[s] := \begin{cases} 1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0, \end{cases} \tag{4.1}$$

then, we can define

$$N_1(t) := E_1 - E(t) + \epsilon\int_0^L \mathrm{Sgn}\left[y^0(L)\right]y^0 y_x(t)y_t(t)\,dx + \epsilon\left|y^0(L)\right|\int_0^L y_t(t)y(t)\,dx + \frac{1}{2}\epsilon\left|y^0(L)\right|\|y(t)\|_2^2, \tag{4.2}$$

where $\epsilon$ is a positive constant which satisfies

$$N_1(0) = E_1 - E(0) + \epsilon\int_0^L \mathrm{Sgn}\left[y^0(L)\right]y^0 y_x^0 y^1\,dx + \epsilon\left|y^0(L)\right|\int_0^L y^1 y^0\,dx + \frac{1}{2}\epsilon\left|y^0(L)\right|\|y^0\|_2^2 > 0 \tag{4.3}$$

and

$$0 < \epsilon < \frac{1}{M^2 + \mathcal{M}}. \tag{4.4}$$

**Proof of Theorem 2.1.** Let

$$\omega(t) := \int_0^L \mathrm{Sgn}\left[y^0(L)\right]y^0 y_x(t)y_t(t)\,dx.$$

By taking a derivative of $\omega(t)$, we get

$$\frac{d}{dt}\omega(t) = \frac{|y^0(L)|}{2}\big|y_t(L,t)\big|^2 + \frac{|y^0(L)|}{2}\big[y(L,t) + \big|y_t(L,t)\big|^{m-1}y_t(L,t)\big]^2 + \frac{|y^0(L)|}{p+1}\big|y(L,t)\big|^{p+1} - \frac{1}{2}\int_0^L \operatorname{Sgn}\big[y^0(L)\big]y_x^0\big|y_t(t)\big|^2\,dx - \frac{1}{2}\int_0^L \operatorname{Sgn}\big[y^0(L)\big]y_x^0\big|y_x(t)\big|^2\,dx - \frac{1}{p+1}\int_0^L \operatorname{Sgn}\big[y^0(L)\big]y_x^0\big|y(t)\big|^{p+1}\,dx - \omega(t). \tag{4.5}$$

Using (4.5) and taking a derivative of $N_1(t)$, we have

$$\frac{d}{dt}N_1(t) \geqslant \big|y_t(L,t)\big|^{m+1} + \big\|y_t(t)\big\|_2^2 + \epsilon\Bigg\{\frac{|y^0(L)|}{2}\big[y(L,t) + \big|y_t(L,t)\big|^{m-1}y_t(L,t)\big]^2 + \frac{|y^0(L)|}{2}\big|y_t(L,t)\big|^2 + \frac{|y^0(L)|}{p+1}\big|y(L,t)\big|^{p+1} - \frac{1}{2}\int_0^L \big|y_x^0\big|\big|y_t(t)\big|^2\,dx - \frac{1}{2}\int_0^L \big|y_x^0\big|\big|y_x(t)\big|^2\,dx - \frac{1}{p+1}\int_0^L \big|y_x^0\big|\big|y(t)\big|^{p+1}\,dx - \omega(t)\Bigg\} + \epsilon\big|y^0(L)\big|\big[\big\|y_t(t)\big\|_2^2 - \big|y(L,t)\big|^2 - \big\|y_x(t)\big\|_2^2 - \big|y_t(L,t)\big|^{m-1}y_t(L,t)y(L,t) + \big\|y(t)\big\|_{p+1}^{p+1}\big]. \tag{4.6}$$

By Young's inequality, it follows that

$$\big|\epsilon\omega(t)\big| \leqslant \frac{M^2\epsilon}{2}\big\|y_t(t)\big\|_2^2 + \frac{\epsilon}{2}\big\|y_x(t)\big\|_2^2. \tag{4.7}$$

Combining (4.6) and (4.7), we conclude that

$$\frac{d}{dt}N_1(t) \geqslant \left[1 - \frac{M^2\epsilon}{2} - \frac{\mathcal{M}\epsilon}{2} + \big|y^0(L)\big|\epsilon\right]\big\|y_t(t)\big\|_2^2 + \frac{\epsilon|y^0(L)|}{2}\big|y_t(L,t)\big|^{2m} + \frac{\epsilon|y^0(L)|}{p+1}\big|y(L,t)\big|^{p+1} - \epsilon\left[\frac{\mathcal{M}}{2} + \frac{1}{2} + \big|y^0(L)\big|\right]\big\|y_x(t)\big\|_2^2 - \epsilon\left[\frac{\mathcal{M}}{2} + \frac{1}{2} + \big|y^0(L)\big|\right]\big|y(L,t)\big|^2 + \epsilon\left[\big|y^0(L)\big| - \frac{\mathcal{M}}{p+1}\right]\big\|y(t)\big\|_{p+1}^{p+1}. \tag{4.8}$$

Using (1.2), (2.1) and (4.8), we get

$$\frac{d}{dt}N_1(t) \geqslant \left[1 - \frac{M^2\epsilon}{2} - \frac{\mathcal{M}\epsilon}{2} + \big|y^0(L)\big|\epsilon\right]\big\|y_t(t)\big\|_2^2 + \frac{\epsilon|y^0(L)|}{2}\big|y_t(L,t)\big|^{2m} + \frac{\epsilon|y^0(L)|}{p+1}\big|y(L,t)\big|^{p+1} + \epsilon\big[\mathcal{M} + 1 + 2\big|y^0(L)\big|\big]\big[E_1 - E(t)\big] - \epsilon\big[\mathcal{M} + 1 + 2\big|y^0(L)\big|\big]E_1 + \epsilon\left[\frac{p-1}{p+1}\big|y^0(L)\big| - \frac{2\mathcal{M}+1}{p+1}\right]\big\|y(t)\big\|_{p+1}^{p+1}. \tag{4.9}$$

From (2.4) we have

$$\frac{p-1}{p+1}\big|y^0(L)\big| - \frac{2\mathcal{M}+1}{p+1} \geqslant \frac{\mathcal{M} + 1 + 2|y^0(L)|}{2}\cdot\frac{p-1}{p+1}\eta(\lambda). \tag{4.10}$$

It follows from (4.4), (4.9), (4.10) and Lemma 3.1 that

$$\frac{d}{dt}N_1(t) \geqslant \gamma_1\big[\big|y_t(L,t)\big|^{2m} + \big|y(L,t)\big|^{p+1}\big], \tag{4.11}$$

where

$$\gamma_1 = \frac{\epsilon |y^0(L)|}{p+1}. \tag{4.12}$$

In order to use Lemma 3.3 we have to verify (3.14). Indeed, it follows from the Cauchy–Schwarz inequality and the Sobolev embedding that

$$N_1(t) \leqslant C\big[\|y_t(t)\|_2^2 + \|y_x(t)\|_2^2 + 2E_1\big]. \tag{4.13}$$

Making use of Lemma 3.1 we get

$$\|y_t(t)\|_2^2 + \|y_x(t)\|_2^2 \leqslant 2E_1 + \frac{2}{p+1}\|y(t)\|_{p+1}^{p+1} \leqslant C\|y(t)\|_{p+1}^{p+1}. \tag{4.14}$$

That is

$$N_1(t) \leqslant C\|y(t)\|_{p+1}^{p+1}. \tag{4.15}$$

From (4.3), (4.11) and (4.15), making use of Lemma 3.3, we complete the proof. □

## 5. The proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2. Define

$$N_2(t) := E_1 - E(t) + \varepsilon\int_0^L x y_x(t) y_t(t)\,dx + L\varepsilon\int_0^L y_t(t) y(t)\,dx + \frac{L}{2}\varepsilon\|y(t)\|_2^2, \tag{5.1}$$

where $\varepsilon$ is a positive constant which satisfies

$$N_2(0) = E_1 - E(0) + \varepsilon\int_0^L x y_x^0 y^1\,dx + L\varepsilon\int_0^L y^1 y^0\,dx + \frac{L}{2}\varepsilon\|y^0\|_2^2 > 0 \tag{5.2}$$

and

$$0 < \varepsilon < \frac{1}{L^2+1}. \tag{5.3}$$

**Proof of Theorem 2.2.** Let

$$\phi(t) := \int_0^L x y_x(t) y_t(t)\,dx.$$

By taking a derivative of $\phi(t)$, we get

$$\begin{aligned}\frac{d}{dt}\phi(t) = {}& \frac{L}{2}\big|y_t(L,t)\big|^2 + \frac{L}{2}\big[y(L,t) + \big|y_t(L,t)\big|^{m-1} y_t(L,t)\big]^2 + \frac{L}{p+1}\big|y(L,t)\big|^{p+1} \\ & - \frac{1}{2}\int_0^L \big|y_t(t)\big|^2\,dx - \frac{1}{2}\int_0^L \big|y_x(t)\big|^2\,dx - \frac{1}{p+1}\int_0^L \big|y(t)\big|^{p+1}\,dx - \phi(t).\end{aligned} \tag{5.4}$$

Using (5.4) and taking a derivative of $N_2(t)$, we have

$$\begin{aligned}\frac{d}{dt}N_2(t) = {}& \big|y_t(L,t)\big|^{m+1} + \|y_t(t)\|_2^2 + \varepsilon\bigg\{\frac{L}{2}\big|y_t(L,t)\big|^2 + \frac{L}{2}\big[y(L,t) + \big|y_t(L,t)\big|^{m-1} y_t(L,t)\big]^2 \\ & + \frac{L}{p+1}\big|y(L,t)\big|^{p+1} - \frac{1}{2}\|y_t(t)\|_2^2 - \frac{1}{2}\|y_x(t)\|_2^2 - \frac{1}{p+1}\|y(t)\|_{p+1}^{p+1} - \phi(t)\bigg\} \\ & + L\varepsilon\big[\|y_t(t)\|_2^2 - \big|y(L,t)\big|^2 - \|y_x(t)\|_2^2 - \big|y_t(L,t)\big|^{m-1} y_t(L,t) y(L,t) + \|y(t)\|_{p+1}^{p+1}\big].\end{aligned} \tag{5.5}$$

By Young's inequality, it follows that

$$\left|\varepsilon\phi(t)\right| \leqslant \frac{L^2\varepsilon}{2}\left\|y_t(t)\right\|_2^2 + \frac{\varepsilon}{2}\left\|y_x(t)\right\|_2^2. \tag{5.6}$$

It follows from (5.5) and (5.6) that

$$\begin{aligned}\frac{d}{dt}N_2(t) \geqslant {} & \left(1 - \frac{L^2\varepsilon}{2} - \frac{\varepsilon}{2} + L\varepsilon\right)\left\|y_t(t)\right\|_2^2 + \frac{\varepsilon L}{2}\left|y_t(L,t)\right|^{2m} + \frac{\varepsilon L}{p+1}\left|y(L,t)\right|^{p+1} \\ & - \varepsilon(1+L)\left\|y_x(t)\right\|_2^2 - \varepsilon(1+L)\left|y(L,t)\right|^2 + \varepsilon\left(L - \frac{1}{p+1}\right)\left\|y(t)\right\|_{p+1}^{p+1}.\end{aligned} \tag{5.7}$$

Using (1.2), (2.1) and (5.7), we get

$$\begin{aligned}\frac{d}{dt}N_2(t) \geqslant {} & \left(1 - \frac{L^2\varepsilon}{2} - \frac{\varepsilon}{2} + L\varepsilon\right)\left\|y_t(t)\right\|_2^2 + \frac{\varepsilon L}{2}\left|y_t(L,t)\right|^{2m} + \frac{\varepsilon L}{p+1}\left|y(L,t)\right|^{p+1} \\ & + 2\varepsilon(1+L)\left[E_1 - E(t)\right] - \varepsilon(1+L)2E_1 + \varepsilon\left(\frac{p-1}{p+1}L - \frac{3}{p+1}\right)\left\|y(t)\right\|_{p+1}^{p+1}.\end{aligned} \tag{5.8}$$

Thanks our choice (2.7) of $L$, it follows that

$$\frac{p-1}{p+1}L - \frac{3}{p+1} \geqslant (L+1)\cdot\frac{p-1}{p+1}\eta(\lambda). \tag{5.9}$$

It follows from (5.3), (5.8), (5.9) and Lemma 3.1 that

$$\frac{d}{dt}N_2(t) \geqslant \gamma_2\left[\left|y_t(L,t)\right|^{2m} + \left|y(L,t)\right|^{p+1}\right], \tag{5.10}$$

where

$$\gamma_2 = \frac{\varepsilon L}{p+1}. \tag{5.11}$$

In order to use Lemma 3.3 we have to verify (3.14). Indeed, it follows from the Cauchy–Schwarz inequality and the Sobolev embedding that

$$N_2(t) \leqslant C\left[\left\|y_t(t)\right\|_2^2 + \left\|y_x(t)\right\|_2^2 + 2E_1\right]. \tag{5.12}$$

Making use of Lemma 3.1 we get

$$\left\|y_t(t)\right\|_2^2 + \left\|y_x(t)\right\|_2^2 \leqslant 2E_1 + \frac{2}{p+1}\left\|y(t)\right\|_{p+1}^{p+1} \leqslant C\left\|y(t)\right\|_{p+1}^{p+1}. \tag{5.13}$$

That is

$$N_2(t) \leqslant C\left\|y(t)\right\|_{p+1}^{p+1}. \tag{5.14}$$

From (5.2), (5.10) and (5.14), making use of Lemma 3.3, we complete the proof. □

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