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# λ′-optimal digraphs ☆

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ABSTRACT

Restricted arc-connectivity is a more refined network reliability index than arc-connectivity. In this paper, we first introduce the concept of minimum arc-degree and show for many digraphs, the minimum arc-degree is an upper bound on the restricted arc-connectivity. Next we call a strongly connected digraph a $\lambda'$-optimal digraph if its restricted arc-connectivity is equal to its minimum arc-degree. Finally, we give some sufficient conditions for a digraph to be $\lambda'$-optimal.



## 1. Terminology and introduction

For graph-theoretical terminology and notation not defined here we follow [4]. We only consider finite (di)graphs without loops and multiple edges (arcs). Let $G$ be a graph. The degree $d(v)$ of a vertex $v$ in $G$ is the number of vertices adjacent to $v$ and the edge degree $\xi(e)$ of an edge $e = uv$ in $G$ is $d(u) + d(v) - 2$. Let $\xi = \xi(G)$ denote the minimum edge degree in $G$. Let $D = (V, A)$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. For a vertex $v$ in $D$, its out-neighborhood is $N^+(v) = N_D^+(v) = \{u \in V(D)\colon vu \in A(D)\}$, its out-degree is $d^+(v) = d_D^+(v) = |N^+(v)|$. The minimum out-degree of $D$ is $\delta^+(D) = \min\{d^+(x)\colon x \in V(D)\}$. The in-neighborhood $N^-(v)$, the in-degree $d^-(v)$ of $v$ and the minimum in-degree $\delta^-(D)$ of $D$ are defined analogously. The minimum degree of $D$ is $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. For a pair $X, Y$ of nonempty vertex sets of $D$, we define $(X, Y) = \{xy \in A(D)\colon x \in X, y \in Y\}$. If $Y = \overline{X} = V(D) \setminus X$, we write $\partial^+(X)$ or $\partial^-(Y)$ instead of $(X, Y)$. Usually, abbreviate $\partial^+(\{x\})$ and $\partial^-(\{x\})$ to $\partial^+(x)$ and $\partial^-(x)$, respectively. Clearly, $d^+(x) = |\partial^+(x)|$ and $d^-(x) = |\partial^-(x)|$. If $C_g = u_1u_2 \ldots u_g u_1$ is a shortest cycle of $D$, then let $\xi(C_g) = \min\{\sum_{i=1}^{g} d^+(u_i) - g, \sum_{i=1}^{g} d^-(u_i) - g\}$, and $\xi(D) = \min\{\xi(C_g)\colon C_g$ is a shortest cycle of $D\}$.

A processor interconnection network or a communications network is conveniently modeled by a graph $D = (V, E)$ or a digraph $D = (V, A)$, in which the vertex set $V$ corresponds to processors or switching elements, and the edge set $E$ or the arc set $A$ corresponds to communication links. One fundamental consideration in the design of networks is reliability. When studying network reliability, one often considers a network model [6] whose vertices are perfectly reliable while edges or arcs may fail independently with the same probability $\rho \in (0, 1)$. An edge (arc)-cut of a (strongly) connected (di)graph $D$ is a set of edges (arcs) whose removal disconnects $D$. The edge (arc)-connectivity $\lambda = \lambda(D)$ is defined as the minimum cardinality over all edge (arc)-cuts of $D$. Let $m_i$ be the number of edge (arc)-cuts of size $i$. Then the probability of $D$ being (strongly) connected is

$$R(D;\rho) = 1 - \sum_{i=\lambda}^{\varepsilon} m_i \rho^i (1-\rho)^{\varepsilon - i},$$

where $\varepsilon$ is the number of edges (arcs) of $D$. The polynomial $R(D;\rho)$ is called the all-terminal reliability of $D$. Clearly, the larger $R(D;\rho)$ is, the more reliable the network is. But in general, to determine $R(D;\rho)$, i.e., to determine every $m_i$, is NP-hard [3,9]. When $\rho$ is sufficiently small, the maximum of $R(D;\rho)$ can be obtained by max-

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imizing $\lambda$ first and then minimizing $m_\lambda, m_{\lambda+1}, \ldots, m_\varepsilon$ sequentially [12].

To maximize $\lambda(D)$ and minimize $m_\lambda$, Bauer et al. [5] defined the super-$\lambda$ (di)graphs. In order to estimate more precisely the reliability of networks, Esfahanian and Hakimi [6] introduced the concept of restricted edge-connectivity. A set of edges $S$ in a connected graph $G$ is called a restricted edge cut if $G - S$ is disconnected and contains no isolated vertex. If such an edge cut exists, then the restricted edge connectivity of $G$, denoted by $\lambda' = \lambda'(G)$, is defined to be the minimum number of edges over all restricted edge cuts of $G$. A connected graph $G$ is called $\lambda'$-connected if $\lambda'(G)$ exists. Esfahanian and Hakimi [6] showed that each connected graph $G$ of order $n \geqslant 4$ except a star $K_{1,n-1}$ is $\lambda'$-connected and satisfies $\lambda(G) \leqslant \lambda'(G) \leqslant \xi(G)$. A $\lambda'$-connected graph $G$ is called a $\lambda'$-optimal graph if $\lambda'(G) = \xi(G)$.

Recently, as a generalization of restricted edge-connectivity to digraphs, the concept of restricted arc-connectivity was introduced by Volkmann [11]. Let $D$ be a strongly connected digraph. An arc subset $S$ of $D$ is a restricted arc-cut of $D$ if $D - S$ has a non-trivial strong component $D_1$, that means a strong component with order at least 2, such that $D - V(D_1)$ contains an arc. The restricted arc-connectivity $\lambda'(D)$ is the minimum cardinality over all restricted arc-cuts of $D$. A strongly connected digraph $D$ is called $\lambda'$-connected, if $\lambda'(D)$ exists. In the same paper, Volkmann proved that each strong digraph $D$ of order $n \geqslant 4$ and girth $g = 2$ or $g = 3$ except some families of digraphs is $\lambda'$-connected and satisfies $\lambda(D) \leqslant \lambda'(D) \leqslant \xi(D)$.

Investigations on the restricted edge-connectivity of graphs were made by many authors, for example, by [1,2,7,8,10,13]. However, closely related results on restricted arc-connectivity have received little attention. In this paper, we introduce the concept of $\lambda'$-optimal digraphs, and show some sufficient conditions for a digraph to be $\lambda'$-optimal.

## 2. The minimum arc-degree of a digraph

A digraph $D$ is called acyclic if it has no cycle. Acyclic digraphs form a well studied family of digraphs due to the following important property.

**Proposition 2.1.** *(See [4].) Every acyclic digraph has a vertex of in-degree zero as well as a vertex of out-degree zero.*

The strong components of a digraph $D$ can be labeled $D_1, \ldots, D_t$ such that there is no arc from $D_j$ to $D_i$ unless $j < i$ [4]. We call such an ordering an acyclic ordering of the strong components of $D$.

Let $D$ be a digraph. If $xy$ is an arc with $yx \notin A(D)$, then call $\xi'(xy) = \min\{d^+(x) + d^+(y) - 1, d^-(x) + d^-(y) - 1, d^+(y) + d^-(x), d^+(x) + d^-(y) - 1\}$ the arc-degree of $xy$. If $xy$ is an arc with $yx \in A(D)$, then call $\xi'(xy) = \min\{d^+(x) + d^+(y) - 2, d^-(x) + d^-(y) - 2, d^+(y) + d^-(x) - 1, d^+(x) + d^-(y) - 1\}$ the arc-degree of $xy$. The minimum arc-degree of $D$ is $\xi'(D) = \min\{\xi'(xy)\colon\ xy \in A(D)\}$.

We shall show that for many digraphs, the minimum arc-degree is an upper bound on the restricted arc-connectivity. This requires the following lemma.

**Lemma 2.1.** *Let $D$ be a strongly connected digraph with $\delta^+(D) \geqslant 3$ or $\delta^-(D) \geqslant 3$ and let $xy$ be an arc of $D$. Then $\partial^+(\{x, y\})$, $\partial^-(\{x, y\})$, $\partial^-(x) \cup \partial^+(y)$, and $\partial^+(x) \cup \partial^-(y)$ are restricted arc-cuts of $D$.*

**Proof.** Let $S$ be one of $\partial^+(\{x, y\})$, $\partial^-(\{x, y\})$, $\partial^-(x) \cup \partial^+(y)$, and $\partial^+(x) \cup \partial^-(y)$ and let $D' = D - S$. Clearly, either $x$ and $y$ are two strong components of $D'$ or $D[\{x, y\}]$ is a 2-cycle and so is a strong component of $D'$. Suppose that $D'' = D - \{x, y\}$ is an acyclic digraph. Then, by Proposition 2.1, there exist $u, v \in V(D'')$ such that $d^-_{D''}(u) = 0$ and $d^+_{D''}(v) = 0$. This implies that $d^-_D(u) \leqslant 2$ and $d^+_D(v) \leqslant 2$, contradicting the assumption that $\delta^+(D) \geqslant 3$ or $\delta^-(D) \geqslant 3$. Therefore, $D''$ contains a non-trivial strong component $D_1$. Clearly, $D_1$ is also a strong component of $D'$ and $D - V(D_1)$ contains the arc $xy$. By definition, $S$ is a restricted arc-cut of $D$. The proof is complete. □

**Theorem 2.1.** *Let $D$ be a strongly connected digraph with $\delta^+(D) \geqslant 3$ or $\delta^-(D) \geqslant 3$. Then $D$ is $\lambda'$-connected and satisfies $\lambda'(D) \leqslant \xi'(D)$.*

**Proof.** By Lemma 2.1, for an arbitrary arc $xy$ in $D$, $\partial^+(\{x, y\})$, $\partial^-(\{x, y\})$, $\partial^-(x) \cup \partial^+(y)$, and $\partial^+(x) \cup \partial^-(y)$ are restricted arc-cuts, which implies that $D$ is $\lambda'$-connected and $\lambda'(D) \leqslant \min\{|\partial^+(\{x, y\})|, |\partial^-(\{x, y\})|, |\partial^-(x) \cup \partial^+(y)|, |\partial^+(x) \cup \partial^-(y)|\}$. If $yx \notin A(D)$, then $|\partial^+(\{x, y\})| = d^+(x) + d^+(y) - 1$, $|\partial^-(\{x, y\})| = d^-(x) + d^-(y) - 1$, $|\partial^-(x) \cup \partial^+(y)| = d^-(x) + d^+(y)$, and $|\partial^+(x) \cup \partial^-(y)| = d^+(x) + d^-(y) - 1$. If $yx \in A(D)$, then $|\partial^+(\{x, y\})| = d^+(x) + d^+(y) - 2$, $|\partial^-(\{x, y\})| = d^-(x) + d^-(y) - 2$, $|\partial^-(x) \cup \partial^+(y)| = d^-(x) + d^+(y) - 1$, and $|\partial^+(x) \cup \partial^-(y)| = d^+(x) + d^-(y) - 1$. Therefore, $\lambda'(D) \leqslant \xi'(xy)$. Combining this with the arbitrariness of $xy$, it follows that $\lambda'(D) \leqslant \xi'(D)$. □

By definition, $\xi'(D) \leqslant \xi(D)$ for many digraphs $D$, for example, for all the digraphs $D$ with $\delta(D) \geqslant 3$. In this sense, $\xi'(D)$ is a better upper bound on $\lambda'(D)$ than $\xi(D)$. Similar to the definition of $\lambda'$-optimal graphs, a $\lambda'$-connected digraph $D$ is called $\lambda'$-optimal if $\lambda'(D) = \xi'(D)$.

## 3. Sufficient conditions for $\lambda'$-optimal digraphs

It is well known that each minimum arc-cut has the form $\partial^+(X)$, where $X$ is a subset of $V(D)$. The example given below shows that there exist some digraphs without minimum restricted arc-cut of the form $\partial^+(X)$.

**Example 3.1.** Let $H$ be a complete digraph with $V(H) = \{x_1, x_2, \ldots, x_p\}$, where $p \geqslant 4$. The digraph $D'$ is defined as the disjoint union of $H$ and 2 additional vertices $u, v$ such that for each $i = 1, 2, \ldots, p$, $x_i$ dominates $v$ and is dominated by $u$. Let $D = (V(D'), A(D') \cup \{uv, vx_1, x_1u\})$. Clearly, $\{vx_1, x_1u\}$ is a restricted arc-cut of $D$. Let $S$ be a subset of $A(D)$ with $|S| \leqslant 2$. If $S \neq \{vx_1, x_1u\}$, then either $D - S$ is strong or $D - S$ has a strong component with order $|V(D)| - 1$, which implies that $S$ is not a restricted arc-cut of $D$. Therefore, $\{vx_1, x_1u\}$ is the unique minimum restricted arc-cut of $D$, which cannot be written as $\partial^+(X)$ for any $X \subseteq V(D)$.

**Theorem 3.1.** *Let $D$ be a $\lambda'$-connected digraph with $\lambda'(D) \leqslant \xi'(D)$. If $D$ has no minimum restricted arc-cut of the form $\partial^+(X)$, where $X$ is a subset of $V(D)$, then $D$ is $\lambda'$-optimal.*

**Proof.** Let $S$ be a minimum restricted arc-cut of $D$ and let $D_1, D_2, \ldots, D_t$ be an acyclic ordering of the strong components of $D' = D - S$. Since $S$ is a restricted arc-cut, there exists a non-trivial strong component $D_j$ of $D'$. If $D[\bigcup_{i=1}^{j-1} V(D_i)]$ contains an arc, then let $X = \bigcup_{i=j}^{t} V(D_i)$. Noting that there is no arc from $\bigcup_{i=j}^{t} V(D_i)$ to $\bigcup_{i=1}^{j-1} V(D_i)$ in $D'$, we have $\partial^+(X) \subseteq S$. Clearly, $\partial^+(X)$ is also a restricted arc-cut. It follows that $S = \partial^+(X)$, a contradiction. Similarly, if $D[\bigcup_{i=j+1}^{t} V(D_i)]$ contains an arc, then let $X = \bigcup_{i=j+1}^{t} V(D_i)$, we have $S = \partial^+(X)$, a contradiction again. Therefore, both $D[\bigcup_{i=1}^{j-1} V(D_i)]$ and $D[\bigcup_{i=j+1}^{t} V(D_i)]$ are empty, which implies that $D_j$ is the unique non-trivial strong component of $D'$. Combining this with the fact that $S$ is a restricted arc-cut, $D - V(D_j)$ contains an arc with one end, say $u$, in $\bigcup_{i=1}^{j-1} V(D_i)$ and the other, say $v$, in $\bigcup_{i=j+1}^{t} V(D_i)$. Then $\partial^+(v) \cup \partial^-(u) \subseteq S$. It is easy to see that $\partial^+(v) \cup \partial^-(u)$ is a restricted arc-cut and so $S = \partial^+(v) \cup \partial^-(u)$. It follows that $\lambda'(D) = |S| = |\partial^+(v) \cup \partial^-(u)| \geqslant \xi'(D)$. Combining this with the assumption $\lambda'(D) \leqslant \xi'(D)$, we have $\lambda'(D) = \xi'(D)$. Therefore, $D$ is $\lambda'$-optimal. The proof is complete. □

A simple, but very useful sufficient condition for a digraph to be $\lambda'$-optimal is given as follows.

**Corollary 3.1.** *Let $D$ be a strongly connected digraph with $\delta^+(D) \geqslant 3$ or $\delta^-(D) \geqslant 3$ and let $S = \partial^+(X)$ be a minimum restricted arc-cut of $D$. If there exists an arc $x'x''$ in $D[X]$ such that $|N^+(x) \cap \overline{X}| \geqslant 2$ for any $x \in X \setminus \{x', x''\}$ or there exists an arc $y'y''$ in $D[\overline{X}]$ such that $|N^-(y) \cap \overline{X}| \geqslant 2$ for any $y \in \overline{X} \setminus \{y', y''\}$, then $D$ is $\lambda'$-optimal.*

**Proof.** By reason of symmetry we only prove the case that there exists an arc $x'x''$ in $D[X]$ such that for any $x \in X \setminus \{x', x''\}$, $|N^+(x) \cap \overline{X}| \geqslant 2$. The hypotheses imply that $\xi'(D) \leqslant |\partial^+(\{x', x''\})| = |(\{x', x''\}, X \setminus \{x', x''\})| + |(\{x', x''\}, \overline{X})| \leqslant 2|X \setminus \{x', x''\}| + |(\{x', x''\}, \overline{X})| \leqslant \sum_{x \in X \setminus \{x', x''\}} |N^+(x) \cap \overline{X}| + |(\{x', x''\}, \overline{X})| = |(X \setminus \{x', x''\}, \overline{X})| + |(\{x', x''\}, \overline{X})| = |(X, \overline{X})| = |S| = \lambda'(D)$. By Theorem 2.1, we have $\lambda'(D) \leqslant \xi'(D)$. Therefore, $\lambda'(D) = \xi'(D)$, which implies that $D$ is $\lambda'$-optimal. □

**Theorem 3.2.** *Let $D$ be a digraph with order $n \geqslant 4$. If $|N^+(u) \cap N^-(v)| \geqslant 3$ for all pairs of vertices $u, v$ with $uv \notin A(D)$, then $D$ is $\lambda'$-optimal.*

**Proof.** Clearly, $D$ is a strong digraph with $\delta(D) \geqslant 3$. By Theorem 2.1, $D$ is $\lambda'$-connected and satisfies $\lambda'(D) \leqslant \xi'(D)$. Suppose, on the contrary, that $D$ is not $\lambda'$-optimal, that is, $\lambda'(D) < \xi'(D)$. Then, by Theorem 3.1, we can assume that there exists $X \subseteq V(D)$ such that $S = \partial^+(X)$ is a minimum restricted arc-cut. Let $X_i = \{x \in X\colon\ |N^+(x) \cap \overline{X}| = i\}$, $\overline{X}_i = \{y \in \overline{X}\colon\ |N^-(y) \cap X| = i\}$, $i = 0, 1$, and let $X_2 = \{x \in X\colon\ |N^+(x) \cap \overline{X}| \geqslant 2\}$, $\overline{X}_2 = \{y \in \overline{X}\colon\ |N^-(y) \cap X| \geqslant 2\}$.

**Claim 1.** *Either $X_0 = \emptyset$ or $\overline{X}_0 = \emptyset$.*

Assume by way of contradiction that there exist $x \in X_0$ and $y \in \overline{X}_0$. Clearly, $xy \notin A(D)$ and so $|N^+(x) \cap N^-(y)| \geqslant 3$. On the other hand, since $x \in X_0$ and $y \in \overline{X}_0$, we have $N^+(x) \subseteq X$ and $N^-(y) \subseteq \overline{X}$, which implies that $N^+(x) \cap N^-(y) = \emptyset$, a contradiction. Claim 1 follows.

Without loss of generality, assume $X_0 = \emptyset$ and let $D_1, D_2, \ldots, D_t$ be an acyclic ordering of the strong components of $D' = D[\overline{X}]$.

**Claim 2.** $|X| \geqslant 3$.

Suppose that $D[X]$ contains no arcs. Then by the definition of restricted arc-cuts, we have $t \geqslant 2$. For any $y_1 \in V(D_1)$, $y_t \in V(D_t)$, we have $y_t y_1 \notin A(D)$ and so $|N^+(y_t) \cap N^-(y_1)| \geqslant 3$. On the other hand, since $y_1 \in V(D_1)$ and $y_t \in V(D_t)$, we have $N^-(y_1) \subseteq V(D_1) \cup X$ and $N^+(y_t) \subseteq V(D_t) \cup X$ and so $N^-(y_1) \cap N^+(y_t) \subseteq X$. Therefore, $|X| \geqslant 3$. Suppose that $D[X]$ contains an arc $x_1x_2$. Then $|X| \geqslant 2$. If $|X| = 2$, then $X = \{x_1, x_2\}$ and so $\lambda'(D) = |S| \geqslant \xi'(x_1x_2) \geqslant \xi'(D)$, contrary to the assumption. Therefore, $|X| \geqslant 3$ again. The proof of Claim 2 is complete.

**Claim 3.** *$D[X]$ contains at least one arc.*

Assume by way of contradiction that $D[X]$ contains no arc. Then, by the definition of restricted arc-cuts, we have $t \geqslant 2$. Suppose that $D_1$ is trivial and let $V(D_1) = \{y_1\}$. Since $D$ is strong, there exists $x \in X$ such that $xy_1 \in A(D)$. Noting that $\partial^+(x) \cup \partial^-(y_1) \subseteq S$, we have that $\lambda'(D) = |S| \geqslant \xi'(xy_1) \geqslant \xi'(D)$, contrary to the assumption. Suppose that $D_t$ is trivial and let $V(D_t) = \{y_t\}$. Since $D$ is strong, there exists $x \in X$ such that $y_t x \in A(D)$. Let $S^* = \partial^+(\{y_t, x\})$. By Lemma 2.1, $S^*$ is a restricted arc-cut of $D$. For any $x', x'' \in X$, by assumption, we have $|N^+(x') \cap N^-(x'')| \geqslant 3$, and so $|N^+(x')| \geqslant 3$. It follows that $|S^*| \leqslant d^+(x) + |X| - 1 < d^+(x) + 3(|X| - 1) \leqslant |S|$, which is contrary to the minimality of $S$. Therefore, both $D_1$ and $D_t$ are not trivial. This implies that $S' = (V \setminus (V(D_1)), V(D_1)) = (X, V(D_1))$ is a restricted arc-cut of $D$. Noting that $S' \subseteq S$, it follows that $S' = S$ from the minimality of $S$. Let $y_t \in V(D_t)$. Then for any $y \in V(D_1)$, we have $y_t y \notin A(D)$. By assumption, $|N^+(y_t) \cap N^-(y)| \geqslant 3$. Combining this with the fact that $N^+(y_t) \cap N^-(y) \subseteq X$, we have $|N^-(y) \cap X| \geqslant 3$ and so $|N^-(y) \cap (V \setminus (V(D_1))| \geqslant 3$. Since $D_1$ is non-trivial, there exists an arc $y_1 y_1'$ in $D_1$. By Corollary 3.1, $D$ is $\lambda'$-optimal, a contradiction completing the proof of Claim 3.

If $X = X_2$, then, by Claim 3 and Corollary 3.1, $D$ is $\lambda'$-optimal, a contradiction. So we have

**Claim 4.** $X_1 \neq \emptyset$.

Let $x_1 \in X_1$ and let $N^+(x_1) \cap \overline{X} = \{y_1\}$.

**Claim 5.** $\overline{X} \setminus \{y_1\} \subseteq \overline{X}_2$.

By the definition of $x_1$, for any $y \in \overline{X} \setminus \{y_1\}$, $x_1 y \notin A(D)$. By assumption, $3 \leqslant |N^+(x_1) \cap N^-(y)| = |N^+(x_1) \cap N^-(y) \cap X| + |N^+(x_1) \cap N^-(y) \cap \overline{X}| \leqslant |N^-(y) \cap X| + |N^+(x_1) \cap \overline{X}| = |N^-(y) \cap X| + 1$, which implies that $|N^-(y) \cap X| \geqslant 2$.

**Claim 6.** $\overline{X}_0 = \emptyset$.

Since $x_1 \in N^-(y_1) \cap X$, we have $y_1 \notin \overline{X}_0$. By Claim 5, we have $y \in \overline{X}_2$ for any $y \in \overline{X} \setminus \{y_1\}$. Therefore, $\overline{X}_0 = \emptyset$.

Similarly, we have that $|\overline{X}| \geqslant 3$, $D[\overline{X}]$ contains at least one arc, and

**Claim 7.** $\overline{X}_1 \neq \emptyset$.

By Claims 5 and 7, we have

**Claim 8.** $\overline{X}_1 = \{y_1\}$.

Let $x \in X \setminus \{x_1\}$. Then $xy_1 \notin A(D)$. By assumption, $3 \leqslant |N^+(x) \cap N^-(y_1)| = |N^+(x) \cap N^-(y_1) \cap X| + |N^+(x) \cap N^-(y_1) \cap \overline{X}| \leqslant |N^-(y_1) \cap X| + |N^-(y_1) \cap \overline{X}| = 1 + |N^-(y_1) \cap \overline{X}|$, which implies that $|N^-(y_1) \cap \overline{X}| \geqslant 2$. Therefore, there exists $y \in \overline{X}$ such that $yy_1 \in A(D)$. By Corollary 3.1 and Claim 5, $D$ is $\lambda'$-optimal, a contradiction. The proof of Theorem 3.2 is complete. □

**Corollary 3.2.** *(See [7].) Let $G$ be a $\lambda'$-connected graph. If $|N(u) \cap N(v)| \geqslant 3$ for all pairs $u$, $v$ of nonadjacent vertices, then $G$ is $\lambda'$-optimal.*

**Proof.** Let $D = D(G)$ be the digraph obtained from $G$ by replacing each edge of $G$ by a pair of two mutually opposite oriented arcs. Then $|N_D^+(u) \cap N_D^-(v)| \geqslant 3$ for all pairs of vertices $u, v$ with $uv \notin A(D)$. By Theorem 3.2, $\lambda'(D) = \xi'(D)$. Clearly, $\xi(G) = \xi'(D)$. It was pointed out in the proof of Corollary 2.2 of [11] that $\lambda'(G) = \lambda'(D)$. It follows that $\lambda'(G) = \xi(G)$. The proof is complete. □

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